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LETTER TO THE EDITOR

Supersymmetry and the Dirac equation of a neutral particle with an anomalous magnetic moment in a central electrostatic field

V V Semenov

Institute of Spectroscopy, USSR Academy of Sciences, Troitsk, Moscow Region 142092, USSR

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Abstract. It is shown that the Dirac equation of a neutral particle with an anomalous magnetic moment in an arbitrary central electrostatic field possesses the structure of supersymmetric quantum mechanics. It is found that only the global behaviour of the electrostatic potential determines an existence of the bound states in a system.

Supersymmetric quantum mechanics (SUSYQM), proposed by Witten [1], has been extensively studied in recent years in many aspects. One of them is the relationship between SUSYQM and the Dirac equation. Ui [2] has shown that for $(1+2)$-dimensional spacetime the Dirac equation of a fermion interacting with any gauge field can be brought into the framework of SUSYQM. Later Gamboa and Zanelli [3] generalised Ui's results to any number of dimensions. The relationship between SUSYQM and the Dirac equation in a certain field was studied in many works. In particular, Sukumar [4] has shown that the Dirac equation in a central Coulomb field has an SUSYQM structure and, using SUSYQM methods, has found the bound state energies and eigenfunctions.

In this letter we consider a connection between SUSYQM and the Dirac particle interacting in a non-minimal way with an external field. Namely, we consider a relativistic neutral particle with spin-$\frac{1}{2}$ in an arbitrary central electrostatic field. The magnetic moment for a neutral particle is the anomalous one, so the Dirac equation for this particle includes interaction with the electromagnetic field only in the Pauli's form [5] and looks as follows:

$$(\gamma^\mu p_\mu - m)\psi = \tfrac{1}{2}\mathrm{i}\mu\sigma^{\mu\nu}F_{\mu\nu}\psi \tag{1}$$

where $c = \hbar = 1$, μ is an anomalous magnetic moment, $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and the Dirac matrices γ^μ are chosen in standard representation.

The central electrostatic field can be presented as $A_0 = \varphi(r)$, $\boldsymbol{A} = 0$. Therefore, the field strength is given by $\boldsymbol{E} = -\mathrm{grad}\,\varphi(r) = -(\boldsymbol{r}/r)(\mathrm{d}\varphi(r)/\mathrm{d}r)$ and, consequently, the Hamiltonian of a particle in this field has the form

$$H = (\boldsymbol{\alpha}\cdot\boldsymbol{p}) + \beta m + \mathrm{i}\mu\beta(\boldsymbol{\alpha}\cdot\boldsymbol{E}). \tag{2}$$

Using the Dirac operator $K = \beta((\boldsymbol{\sigma}\cdot\boldsymbol{L})+1)$ ($\boldsymbol{L} = [\boldsymbol{r}\times\boldsymbol{p}]$ is an orbital angular momentum of the particle) the Hamiltonian of the system (2) can be written as follows:

$$H = \alpha_r p_r + \beta m + \mathrm{i}\alpha_r\beta\frac{K}{r} + \mathrm{i}\mu\beta\alpha_r E_r \tag{3}$$

where $p_r = -\mathrm{i}[(\mathrm{d}/\mathrm{d}r)+(1/r)]$, $E_r = -\mathrm{d}\varphi/\mathrm{d}r = -\varphi'$ and for matrix $\alpha_r = (\boldsymbol{r}\cdot\boldsymbol{\alpha})/r$ we choose the representation:

$$\alpha_r = \begin{vmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{vmatrix}$$

with $\sigma_r = (\boldsymbol{r}\cdot\boldsymbol{\sigma})/r$.

The Hamiltonian (3) commutes with the Dirac operator K and with the projection of the angular momentum operator J_z, so the wavefunction in a central electrostatic field can be presented as

$$\psi_{Ekj_z} = \begin{vmatrix} u(r)\chi_{kj_z}(\varphi,\theta) \\ v(r)\chi_{-kj_z}(\varphi,\theta) \end{vmatrix} = \frac{1}{r}\begin{vmatrix} f(r)\chi_{kj_z}(\varphi,\theta) \\ g(r)\chi_{-kj_z}(\varphi,\theta) \end{vmatrix} \qquad (4)$$

where angular functions $\chi_{kj_z}(\varphi,\theta)$ and $\chi_{-kj_z}(\varphi,\theta)$ are usual spherical spinors. For the determination of the $f(r)$ and $g(r)$ we obtain the equations

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}-\mu\varphi'(r)\right)g(r) = (E-m)f(r)$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}-\mu\varphi'(r)\right)f(r) = (E+m)g(r). \qquad (5)$$

One may consider (5) as the relations between two components of the supersymmetric wavefunction. So, we shall consider the $f(r)$ to represent the 'fermionic' sector and the $g(r)$ to represent the 'bosonic' sector of eigenfunctions of the supersymmetric Hamiltonian. Thus, (5) may be considered as a representation of a transformation between 'fermionic' and 'bosonic' sectors introduced by charges Q^+ and Q^-. Taking this point of view, one can easily note that the nil-potent conserving charge operators Q^+ and Q^- are

$$Q^- = \begin{vmatrix} 0 & 0 \\ A^+ & 0 \end{vmatrix} \qquad Q^+ = \begin{vmatrix} 0 & A^- \\ 0 & 0 \end{vmatrix} \qquad (6)$$

where $A^{\pm} = [\pm(\mathrm{d}/\mathrm{d}r) - k/r - \mu\varphi'(r)]$ and the supersymmetric Hamiltonian is of the standard form

$$H_{\mathrm{ss}} = \{Q^-, Q^+\} = \begin{vmatrix} A^-A^+ & 0 \\ 0 & A^+A^- \end{vmatrix}. \qquad (7)$$

Consequently, we can write down the eigenvalue equations as

$$A^-A^+f(r) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + W^2 - W'\right)f(r) = (E^2-m^2)f(r) \qquad (8a)$$

$$A^+A^-g(r) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + W^2 + W'\right)g(r) = (E^2-m^2)g(r) \qquad (8b)$$

where $W = -(k/r) - \mu\varphi'(r)$.

Equations (8*a*) and (8*b*) have identical energy spectra except for the ground state. In the case where SUSY is unbroken, the ground-state wavefunction is non-degenerate and is defined by one of the next equations

$$A^+f_0(r) = 0 \qquad (9a)$$

$$A^-g_0(r) = 0. \qquad (9b)$$

The ground state wavefunction corresponds to the normalisable solution of these equations. In our case (9*a*) and (9*b*) are evidently integrated and those solutions are

$$f_0(r) = C_1 r^k \exp(\mu\varphi(r)) \tag{10a}$$

$$g_0(r) = C_2 r^{-k} \exp(-\mu\varphi(r)) \tag{10b}$$

with the ground-state energy eigenvalue $E_0^2 = m^2$.

Thus, it follows from (10*a*) and (10*b*) that only the global properties of the electrostatic potential (not its specific form) define the presence of the ground state. In other words, the asymptotic behaviour of the electrostatic potential at zero and infinity defines existence of the normalisable solution.

So far we have considered eigenstates of the Dirac operator K with positive k. However, the obtained results are conserved for the states with negative k replacing k by $-|k|$ in all the formulae. For example, (9*a*) and (9*b*) become

$$q_0(r) = C_3 r^{-|k|} \exp(\mu\varphi(r)) \tag{11a}$$

$$p_0(r) = C_4 r^{|k|} \exp(-\mu\varphi(r)). \tag{11b}$$

Note, that for the bound states of the initial system with the Hamiltonian (3) to exist, apart from the normalisability of $(1/r)f_0$ (or $(1/r)g_0$) another function g_0 (or f_0) should vanish. It is because f_0 and g_0 are connected due to system (5) and both these functions are parts of solution (4).

As an example let us consider the case of the Coulomb field: $\mu\varphi(r) = \mu Ze/r > 0$. From (10*a*), (10*b*), (11*a*) and (11*b*) one can see that the normalisable solution is

$$g_0(r) = r^{-k} \exp\left(-\mu \frac{Ze}{r}\right)$$

with energy $E_0^2 = m^2$. Further, from (5) we find that at $E_0 = m f_0$ is not normalisable and at $E_0 = -m, f_0 = 0$. Thus, the bound state of the system with the Hamiltonian (3) for the Coulomb field exists at $E_0 = -m$ and is given by

$$\psi_{Ekj_z} = C \begin{vmatrix} 0 \\ r^{-(k+1)} \exp\left(-\mu \dfrac{Ze}{r}\right) \chi_{-kj_z}(\varphi, \theta) \end{vmatrix}.$$

To summarise, here we have shown that the Dirac particle with an anomalous magnetic moment in a central symmetric electrostatic field can be considered in the framework of SUSYQM.

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